

Proposal for Generic Size of Large Extra Dimensions

Satoshi Matsuda[†]

*Department of Physics, Graduate School of Science, Kyoto University
Yoshida-Nihonmatsu, Sakyo, Kyoto 606-8501, Japan*

and

Shigenori Seki[‡]

*Theoretical Physics Laboratory
The Institute of Physical and Chemical Research (RIKEN)
2-1 Hirosawa, Wako 351-0198, Japan*

Abstract

In order to resolve the hierarchy problem, large extra dimensions have been introduced, and it has been suggested that the size of extra dimensions is sub-millimeter. On the other hand, we assume in this paper that the cosmological constant comes from the Casimir energy of extra dimensions and estimate the size of extra dimensions in terms of the value of the cosmological constant discovered by the recent WMAP observations. We demonstrate that this size is consistent with the one derived from the hierarchy problem and propose that there may be a generic size of large extra dimensions when the number of extra dimensions is equal to two.

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[†] matsuda@phys.h.kyoto-u.ac.jp

[‡] seki@postman.riken.go.jp

1. Introduction

Large extra dimensions have been introduced in Refs.[1,2] in order to resolve the hierarchy problem. The interesting possibility of their existence has stimulated much theoretical [3–5] and experimental research [6–8].

On the other hand, the origin of the cosmological constant has not been made clear so far. This is one of the most important problems in physics. Recently the WMAP project [9] has determined accurately several parameters of our universe, for example, the age of universe, the density of dark matter and so on.

In section 2 of this paper we calculate the Casimir energy derived from extra dimensions. In section 3 we assume that the Casimir energy can be identified with the cosmological constant, and estimate the size of extra dimensions by the use of the observed value of the cosmological constant. Section 4 is devoted to the conclusions, where we compare the size of extra dimensions corresponding to the cosmological constant problem with the one corresponding to the hierarchy problem.

2. Casimir energy and cosmological constant

We consider the d dimensional compact extra space which is the product of d spheres, $(S^1)^{\otimes d}$. We set that the size of every S^1 is R . The Kaluza-Klein modes of gravitons are generated from this space and contribute to the vacuum energy in our four dimensional space-time. The vacuum energy is obtained as the Casimir energy derived from the extra space, and is given by

$$E = \left[\frac{(2+d)(3+d)}{2} - 1 \right] \frac{1}{2} \left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) \int \frac{d^3 k}{(2\pi)^3} \sqrt{\mathbf{k}^2 + M^2}, \quad (2.1)$$

where $M^2 \equiv 4\pi^2 \sum_{i=1}^d n_i^2 / R^2$ and the factor in front, $(2+d)(3+d)/2 - 1$, represents the number of polarization degrees of freedom for a graviton in $4+d$ dimensions [10]. Since this integration is generally divergent, we use dimensional regularization and Eq.(2.1) becomes

$$E = -\frac{1}{64\pi^2} \left[\frac{(2+d)(3+d)}{2} - 1 \right] \left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) M^4 \left[\ln \left(\frac{\lambda^2}{M^2} \right) + \frac{3}{2} \right]. \quad (2.2)$$

λ is a parameter with mass dimension. We calculate the following infinite summations,

$$\begin{aligned} \left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) \left(\sum_{i=1}^d n_i^2 \right)^2 &= d \left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) n_1^4 + d(d-1) \left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) n_1^2 n_2^2 \\ &= 2d\zeta(0)^{d-1}\zeta(-4) + 4d(d-1)\zeta(0)^{d-2}\zeta(-2)^2 \\ &= 0, \end{aligned} \quad (2.3)$$

where we have used the zeta function equalities $\zeta(-4) = \zeta(-2) = 0$. Using Eq.(2.3), the vacuum energy (2.2) is reduced to

$$E = \frac{\pi^2}{2R^4} \left[\frac{(2+d)(3+d)}{2} - 1 \right] \left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) \frac{d}{dx} N^x \Big|_{x=4}, \quad (2.4)$$

where $N \equiv \sqrt{\sum_{i=1}^d n_i^2}$. In order to take care of a divergence of the right hand side of Eq.(2.4), we have to regularize it. Since it is difficult to calculate it analytically,* let us consider a numerical method. In terms of the Fourier transformation [12], N^x is described as

$$N^x = -\frac{2^x \sin \frac{\pi x}{2}}{\pi^{\frac{d}{2}+1}} \Gamma\left(1 + \frac{x}{2}\right) \Gamma\left(\frac{x+d}{2}\right) \int q^{-x-d} e^{i\mathbf{N} \cdot \mathbf{q}} d\mathbf{q}, \quad (2.5)$$

where $\mathbf{N} \equiv (n_1, \dots, n_d)$, $\mathbf{q} \equiv (q_1, \dots, q_d)$ and $q \equiv |\mathbf{q}|$, while the infinite summation of $\exp(i\mathbf{N} \cdot \mathbf{q})$ is expressed by

$$\left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) e^{i\mathbf{N} \cdot \mathbf{q}} = (2\pi)^d \prod_{i=1}^d \left[\sum_{m_i=-\infty}^{\infty} \delta(q_i - 2\pi m_i) \right]. \quad (2.6)$$

From Eqs.(2.5) and (2.6), we obtain

$$\left(\prod_{i=1}^d \sum_{n_i=-\infty}^{\infty} \right) N^x = -\frac{\sin \frac{\pi x}{2}}{\pi^{1+x+\frac{d}{2}}} \Gamma\left(1 + \frac{x}{2}\right) \Gamma\left(\frac{x+d}{2}\right) \left(\prod_{\substack{i=1 \\ \mathbf{m} \neq \mathbf{0}}}^d \sum_{m_i=-\infty}^{\infty} \right) \left(\sum_{i=0}^d m_i^2 \right)^{-\frac{x+d}{2}}, \quad (2.7)$$

where $\mathbf{m} \equiv (m_1, \dots, m_d)$. Using this equation, the vacuum energy (2.4) becomes

$$E = -\frac{1}{2\pi^{2+\frac{d}{2}} R^4} \left[\frac{(2+d)(3+d)}{2} - 1 \right] \Gamma\left(2 + \frac{d}{2}\right) \mathcal{M}_d, \quad (2.8)$$

$$\mathcal{M}_d \equiv \left(\prod_{\substack{i=1 \\ \mathbf{m} \neq \mathbf{0}}}^d \sum_{m_i=-\infty}^{\infty} \right) \left(\sum_{i=0}^d m_i^2 \right)^{-2-\frac{d}{2}}. \quad (2.9)$$

* In the special cases of d , we can obtain exact results in terms of Epstein zeta functions [11].

The divergence coming from the infinite summations in Eq.(2.4) is thus resolved and Eq.(2.8) converges. Calculating Eq.(2.9) numerically, we obtain

$$\mathcal{M}_1 \approx 1.03693, \quad \mathcal{M}_2 \approx 4.6589, \quad \mathcal{M}_3 \approx 7.46706. \quad (2.10)$$

Substituting (2.10) into Eq.(2.8), the vacuum energies for $d = 1, 2, 3$ become

$$E \approx \frac{1}{R^4} \begin{cases} -0.196993, & d = 1, \\ -1.35231, & d = 2, \\ -3.16082, & d = 3. \end{cases} \quad (2.11)$$

Note that the vacuum energies corresponding to the higher dimensions are similarly calculable. The result for every dimension turns out to be negative.

In the following sections we will suggest that the vacuum energies correspond to the cosmological constant. Since the energies in (2.11) are negative, we can not adopt such identification. So far we have considered the extra space which is $(S^1)^{\otimes d}$, that is, the d dimensional compact space with a periodic boundary condition. On the other hand, if the extra space has a Dirichlet boundary condition, the vacuum energies may become positive. Let us consider the d dimensional compact extra space with the Dirichlet boundary condition in the rest of this section.

Since the wave function becomes a sine function on account of the Dirichlet boundary condition, the vacuum energy is modified from Eq.(2.1) to the form

$$E = \frac{1}{2} \left[\frac{(2+d)(3+d)}{2} - 1 \right] \left(\prod_{i=1}^d \sum_{n_i=1}^{\infty} \right) \int \frac{d^3 k}{(2\pi)^3} \sqrt{\mathbf{k}^2 + M^2}. \quad (2.12)$$

We rewrite this equation as

$$\begin{aligned} E &= \frac{1}{2} \left[\frac{(2+d)(3+d)}{2} - 1 \right] \prod_{i=1}^d \left[\frac{1}{2} \left(\sum_{n_i=-\infty}^{-1} + \sum_{n_i=1}^{\infty} \right) \right] \int \frac{d^3 k}{(2\pi)^3} \sqrt{\mathbf{k}^2 + M^2} \\ &= \frac{1}{2 \cdot 2^d} \left[\frac{(2+d)(3+d)}{2} - 1 \right] \prod_{i=1}^d \left[\sum_{n_i=-\infty}^{\infty} (1 - \delta_{n_i 0}) \right] \int \frac{d^3 k}{(2\pi)^3} \sqrt{\mathbf{k}^2 + M^2} \\ &= \frac{\pi^2}{2^{d+1} R^4} \left[\frac{(2+d)(3+d)}{2} - 1 \right] \prod_{i=1}^d \left[\sum_{n_i=-\infty}^{\infty} (1 - \delta_{n_i 0}) \right] \frac{d}{dx} N^x \Big|_{x=4}. \end{aligned}$$

After numerical calculations, we obtain

$$E \approx \frac{1}{R^4} \begin{cases} -0.196993, & d = 1, \\ 0.0165101, & d = 2, \\ -0.0199402, & d = 3, \\ 0.00149895, & d = 4. \end{cases} \quad (2.13)$$

In the results (2.13), when the numbers of extra dimensions are even, the vacuum energies become positive.

3. The cosmological constant and the size of extra dimensions

Recently the WMAP project has determined the several parameters of our universe [9]. Let us consider the cosmological constant by using the data given by WMAP. Since the Hubble constant is

$$\begin{aligned} H_0 &= h \times 100 \text{ [km} \cdot \text{sec}^{-1} \cdot \text{Mpc}^{-1}] \\ &= 0.7675 \times 10^{-28} \text{ [cm}^{-1}], \end{aligned} \quad (3.1)$$

where $h = 0.71$ (according to the WMAP observation), the critical density of universe becomes

$$\begin{aligned} \rho_c &= \frac{3H_0^2}{8\pi G} \\ &= 5.313 \times 10^3 \text{ [eV} \cdot \text{cm}^{-3}], \end{aligned} \quad (3.2)$$

where G is the gravitation constant. WMAP gives for the total density Ω_{tot} and the dark energy density Ω_Λ ,

$$\begin{aligned} \Omega_{\text{tot}} &= 1.02 \pm 0.02, \\ \Omega_\Lambda &= 0.73 \pm 0.04. \end{aligned} \quad (3.3)$$

Note that the dark energy is the same as the cosmological constant. In terms of the data (3.2) and (3.3), we calculate the cosmological constant to be

$$\begin{aligned} \Lambda &= \rho_c \Omega_{\text{tot}} \Omega_\Lambda \\ &= 3.956 \times 10^3 \text{ [eV} \cdot \text{cm}^{-3}]. \end{aligned} \quad (3.4)$$

We suggest that the cosmological constant is identified with the vacuum energy derived from the extra dimensions.

In the previous section, we have obtained the vacuum energy with the positive values when the number of extra dimensions is even and the extra spaces have the Dirichlet boundary conditions. Let us transform the canonical values (2.13) of the vacuum energy in units of $\hbar c = 1.973269 \times 10^{-5} \text{ [eV} \cdot \text{cm}]$ to the physical values. Then the vacuum energy is calculated to be

$$E \text{ [eV} \cdot \text{cm}^{-3}] = \frac{1}{R^4} \begin{cases} 3.25787 \times 10^{-7}, & d = 2, \\ 2.95782 \times 10^{-8}, & d = 4, \end{cases} \quad (3.5)$$

where the unit of R is centimeter. If we identify the cosmological constant (3.4) with the vacuum energy (3.5), the size R of extra dimensions then becomes

$$R \text{ [cm]} = \begin{cases} 0.003012, & d = 2, \\ 0.001654, & d = 4. \end{cases} \quad (3.6)$$

4. Conclusions

In Ref.[2], the scale of extra dimensions is discussed from the viewpoint of the hierarchy problem. The four dimensional Planck scale M_{pl} is related to the $4 + d$ dimensional one $M_{\text{pl}(4+d)}$ by $M_{\text{pl}}^2 \sim M_{\text{pl}(4+d)}^{2+d} R^d$. If we put $M_{\text{pl}(4+d)} \sim m_{\text{EW}} \sim 1 \text{ [TeV]}$, where m_{EW} is an electro-weak scale, the size of extra dimensions then becomes

$$R \sim 10^{\frac{30}{d}-17} \text{ [cm]}. \quad (4.1)$$

So R has a sub-millimeter size of 10^{-2} [cm] when the number d of extra dimensions is equal to two. If $d = 4$, R has a size of 10^{-10} [cm] .

On the other hand, we have adopted the suggestion that the cosmological constant can be identified with the vacuum energy which is the Casimir energy coming from the space of extra dimensions. By this identification we have calculated the sizes of extra dimensions. We have used the recent results given by the WMAP project. In (3.6), when the number d of extra dimensions equals two, the size R of extra space becomes $3.012 \times 10^{-3} \text{ [cm]}$. When $d = 4$, the size R is $1.654 \times 10^{-3} \text{ [cm]}$.

Comparing these sizes of extra dimensions obtained from the two independent points of view, that is, the hierarchy problem and the cosmological constant, the two sizes for $d = 2$ turn out to be almost consistent. So we propose that there may be the generic size of large extra dimensions for $d = 2$, which is in the range of 10^{-2} – 10^{-3} [cm] .

Finally let us conclude with a brief comment that our proposal also resolves the critical issue often raised where the enormous gap of energy scale in the hierarchy problem is just transmuted into an arbitrarily chosen size of large extra dimensions.

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